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SUPPLEMENTARY INFORMATION

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Translation correlations in anisotropically scattering media

Supplementary material A: Rigorous derivation of the anisotropic memory effect

Here we present a rigorous derivation of the anisotropic memory effect. We prove that the anisotropic memory effect is present for all scattering objects, including ordered or absorbing ones, as long as 1) light propagation is linear, and 2) the directionality of the incident beam is maintained to any extent (i.e., the k-space intensity propagator P_k is not constant across angles).

For generality, we consider the case of continuous fields $E(x_a)$ and $E(x_b)$ along the front and back surfaces of a scattering slab. We discretize this analysis in Supplementary Material B. Wave propagation through any linear medium can be described by a complex transmission function $T_x(x_a, x_b)$ such that,

$$E(x_b) = \int T_x(x_a, x_b)\, E(x_a) \mathrm{d}^2 x_a. \tag{A1}$$

Here, $E(x_a)$ is the incident field on side A, $E(x_b)$ is the transmitted field on side B, x_a and x_b are two-dimensional spatial coordinates in arbitrary planes in front of and behind the sample, respectively.

We now proceed to define the k-space intensity propagator, $P_k(k_b, k_a)$. The k-space intensity propagator gives the average transmitted intensity $I(k_b)$ when the medium is illuminated by a plane wave (with unit power) on side A. By convention, we define the intensity as a function of propagation direction k_b as, $I(k_b) \equiv |E(k_b)|^2$.

First, we construct a truncated plane wave with unit power:

$$E_A(x_a) = \frac{H_A(x_a)}{\sqrt{A}} e^{ik_a \cdot x_a}, \quad H_A(x_a) \equiv \begin{cases} 1 & \text{for } x_a \text{ inside a square area } A \\ 0 & \text{otherwise} \end{cases} \tag{A2}$$

Here, we have $\int_A |E_A(x_a)|^2 \mathrm{d}^2 x_a = 1$. Note that while Eq. (A2)'s wave is truncated to a square area A, we will take the limit of $A \to \infty$ below. Following Eq. (A1), the incident field $E_A(x_a)$ will generate the following transmitted field:

$$E_A(x_b) = \frac{1}{\sqrt{A}} \int T_x(x_a, x_b) H_A(x_a) e^{ik_a \cdot x_a}\, \mathrm{d}^2 x_a. \tag{A3}$$

The resulting k-space intensity propagator is thus defined as the average spectrum intensity at side B, when an incident plane wave with wavevector k_a illuminates side A:

$$P_k(k_b, k_a) \equiv \lim_{A\to\infty} \left\langle \left| \int e^{-ik_b \cdot x_b}\, E_A(x_b)\, \mathrm{d}^2 x_b \right|^2 \right\rangle \tag{A4}$$

$$= \left\langle \frac{1}{A} \iint e^{-ik_b \cdot x_b}\, T_x(x_a, x_b)\, H_A(x_a) e^{ik_a \cdot x_a} \mathrm{d}^2 x_a \mathrm{d}^2 x_b \right.$$

$$\left. \times \iint e^{ik_b \cdot x_{b'}}\, {T_x}^*(x_{a'}, x_{b'})\, H_A(x_{a'}) e^{-ik_a \cdot x_{a'}}\, \mathrm{d}^2 x_{a'} \mathrm{d}^2 x_{b'} \right\rangle$$

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Here, the averaging is performed over many random scatterer configurations. It is direct to show that this definition of P_k is equivalent to the form used in the main text. Next, we apply the coordinate transform $x_{a'} \to x_a - \Delta x_a$, $x_{b'} \to x_b - \Delta x_b$ and reorder Eq. (A4) to find,

$$P_k(k_b, k_a) = \lim_{A\to\infty} \frac{1}{A} \iiiint \langle T_x(x_a, x_b) T_x{}^*(x_a - \Delta x_a, x_b - \Delta x_b)\rangle \tag{A5}$$

$$\times H_A(x_a)\, H_A(x_a - \Delta x_a)\, e^{ik_a\cdot\Delta x_a} e^{-ik_b\cdot\Delta x_b}\, \mathrm{d}^2x_a\, \mathrm{d}^2x_b\, \mathrm{d}^2\Delta x_a\, \mathrm{d}^2\Delta x_b$$

We now define our shift-shift correlation function C_x in the limit $A \to \infty$:

$$C_x(\Delta x_a, \Delta x_b) \equiv \lim_{A\to\infty} \frac{1}{A} \iint H_A(x_a) H_A(x_a - \Delta x_a) \tag{A6}$$

$$\times\langle\, T_x(x_a, x_b)\, T_x{}^*(x_a - \Delta x_a, x_b - \Delta x_b)\rangle\, \mathrm{d}^2x_a\, \mathrm{d}^2x_b,$$

$$= \lim_{A\to\infty} \frac{1}{A} \int_A \int_A \langle T_x(x_a, x_b)\, T_x{}^*(x_a - \Delta x_a, x_b - \Delta x_b)\rangle\, \mathrm{d}^2x_a\, \mathrm{d}^2x_b.$$

Note that Eq. (A6)'s correlation function can always be defined, even when the medium is not shift-invariant. However, defining it only makes sense when the medium is statistically invariant to translations over the area of interest. We may now insert Eq. (A6) into Eq. (A5) to find,

$$P_k(k_b, k_a) = \iint C_x(\Delta x_a, \Delta x_b)\, e^{ik_a\cdot\Delta x_a} e^{-ik_b\cdot\Delta x_b}\, \mathrm{d}^2\Delta x_a\, \mathrm{d}^2\Delta x_b. \tag{A7}$$

Inverting this Fourier transform gives,

$$C_x(\Delta x_a, \Delta x_b) = \iint P_k(k_b, k_a)\, e^{-ik_a\cdot\Delta x_a} e^{ik_b\cdot\Delta x_b}\, \mathrm{d}^2k_a\, \mathrm{d}^2k_b, \tag{A8}$$

which is the continuous equivalent to Eq. (4) in the main manuscript. For convenience, we can change the coordinates of P_k to $k \equiv k_a$ and $\Delta k \equiv k_b - k_a$ and write,

$$C_x(\Delta x_a, \Delta x_b) = \iint P_k'(k; \Delta k)\, e^{i\Delta k\cdot\Delta x_b} e^{-ik\cdot(\Delta x_a - \Delta x_b)} \mathrm{d}^2k\, \mathrm{d}^2\Delta k, \tag{A9}$$

which is still a general expression. Note that in most circumstances $C_x(\Delta x_a, \Delta x_b) \neq 0$, even for $\Delta x_a \neq \Delta x_b$.

Interpretation

In its most general form, Eq. (A9)'s shift-shift correlation function depends on both Δx_a and Δx_b. If we want to calculate the magnitude of the anisotropic memory effect, we evaluate the correlation function at $\Delta x_a = \Delta x_b = \Delta x$ to find that it is simply the Fourier transform of an angle-averaged angular intensity propagator:

$$C_x(\Delta x) = \int \overline{P_\mathrm{k}}(\Delta k)\, e^{i\Delta k\cdot\Delta x}\, \mathrm{d}^2\Delta k. \tag{A10}$$

Here, $\overline{P_k}$ is an angle-averaged k-space intensity propagator that we define as, $\overline{P_k}(\Delta k) \equiv \int P_k'(k; \Delta k)\, \mathrm{d}^2k$. The width of $\overline{P_k}(\Delta k)$ depends on the scattering object's mean free path, anisotropy coefficient, thickness, reflections at the object surface, and possibly other parameters. If a disordered scattering object's width L is much thicker than the transport mean free path for light l, all directionality is lost and $\overline{P_k}(\Delta k)$ will be a constant function across Δk. In this case, the width of $\overline{P_k}(\Delta k)$ will still be limited to $2k_0$, with $k_0 \equiv 2\pi/\lambda$. This finite support of $\overline{P_k}$ gives rise to trivial correlations on the scale of half a wavelength.

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If a disordered scattering object's width L is thinner than one transport mean free path l, some level of directionality is preserved and $\overline{P_k}(\Delta k)$ will be narrower than for the case when $L \gg l$. The narrower the angular intensity propagator's support, the larger the distance over which the sample can be translated while preserving correlations in the speckle at the scatterer's back surface.

Special case: P_k is separable

If Eq. (A9)'s intensity propagator $P_k(k; \Delta k)$ is separable into $P_k^0(k)$ and $P_k^\Delta(\Delta k)$, we may rewrite it in the form,

$$C_x(\Delta x_a, \Delta x_b) = \int P_k^\Delta(\Delta k)\, e^{i\Delta k \cdot \Delta x_b} \mathrm{d}^2\Delta k \int P_k^0(k) e^{-ik\cdot(\Delta x_b - \Delta x_a)}\, \mathrm{d}^2 k, \qquad \text{(A11)}$$

which is the product of the Fourier transform of P_k^Δ with the inverse Fourier transform of P_k^0. In the special case where P_k only depends on the difference between the incident and transmitted wave angle Δk, we find that P_k^0 is constant. Under this condition, Eq. (A11) simplifies to,

$$C_x(\Delta x_a, \Delta x_b) = \delta(\Delta x_b - \Delta x_a) \int P_k(\Delta k) e^{i\Delta k \Delta x_b}\, \mathrm{d}^2\Delta k. \qquad \text{(A12)}$$

In Eq. (A12)'s final form, it is immediately clear that the anisotropic memory effect is the real-space analogy to the traditional memory effect, matching Eq. (5) in the main text.

Conclusion

The anisotropic memory effect exists *for any linear medium*, as long as the directionality of the input waves is maintained to some degree at a chosen output plane (where the 'output' plane could be inside the sample, on the opposite surface, or beyond the opposite surface in free space). The effect of shifting the medium is always robustly defined as the Fourier transform of the k-space intensity propagator, $\overline{P_k}(\Delta k)$. In the C_1 approximation, the intensity-intensity correlation function is simply obtained by taking the absolute square of the field-field correlation C_x in Eq. (A11) or Eq. (A12). Finally, the interested reader may replace all variables and functions above with their respective Fourier conjugates. An analogous, generalized derivation of the traditional memory effect will be the result.

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Supplementary Material B: Discretization into transmission matrices

It is direct to transfer our mathematical findings for the continuous case in Supplementary Material A to the discrete equations introduced in the main text.

First, we may apply Shannon's sampling theorem to discretize the input and output fields, $E(x_a)$ and $E(x_b)$, into vectors. Assuming a 1D geometry, we set the resolution of each vector element at $\delta x_a = \delta x_b = \lambda/2NA$, where the numerical aperture NA is the maximum acceptance angle of light on the input/output surface (which we here assume are equal, for simplicity). Given the area of the input and output surfaces each equal A, then we find the number of spatial modes on each surface as $n = 2NA \cdot A/\lambda$. We may write $E_A(x_a)$ and $E_A(x_b)$ [from Eq. (A2)-(A3)] each as $1{\times}n$ vectors, which are now connected by a discrete $n{\times}n$ transmission matrix, $T_x(x_a, x_b)$.

We may now re-write Eq. (A4) in discrete form as,

$$P_k(k_b) \equiv \langle \left| \mathcal{F}_{1D}^{x_b \to k_b} E_A(x_b) \right|^2 \rangle \tag{B1}$$

where $\mathcal{F}$ is a discrete Fourier transform operator and $P_k(k_b)$ is now one column of our discrete k-space intensity propagator matrix. Eq. (A1) connects $E_A(x_b)$ to the spatial transmission matrix T_x. Thus, we may generate multiple averaged fields, each under plane wave illumination from a unique wavevector k_a, and place their Fourier transform in one row of a matrix:

$$P_k(k_a, k_b) \equiv \langle |T_k(k_a, k_b)|^2 \rangle. \tag{B2}$$

Eq. (B2) is the k-space intensity propagator matrix defined in the main text.

Supplementary Material C: Theoretical prediction of the k-space intensity propagator

Here, we detail one way to connect the anisotropic memory effect's k-space intensity propagator, $P_k(k_a, k_b)$, to scattering material parameters like g and L.

Prior derivations of the traditional memory effect (e.g., [Li1994], [Ber1989]) use the radiative transport equation (RTE) in the diffusion approximation to connect the intensity propagator, $P_x(x_a, x_b)$, to scattering material parameters. The diffusion approximation typically requires a material thickness of several transport mean-free paths. The anisotropic memory effect will be most pronounced, and of greatest use, within one transport mean-free path. Thus, we opt here to follow a different derivation route for the anisotropic memory effect. Our alternative solution more accurately models the transport of light through thin, anisotropic slabs.

Instead of using the diffusion approximation to simplify the RTE, [Ish1978, Kok1997] present an alternative approximation based upon the small-angle approximation (SAA). This limit assumes all light is highly forward scattered (g > 0.8, approximately). Many biological materials of interest, including the tissue tested in our experiment, satisfy this condition.

If we assume the optical setup of interest is cylindrically symmetric, [Kok1997] uses the SAA of the RTE to predict the intensity spectrum across the back surface of a scatterer, when under plane wave illumination, $I_k(k_b)$:

$$I_k(k_b) = \frac{1}{\lambda} \cdot \mathcal{F}^{x \to k_b} \exp\left[-\tau\left(1 - \omega \hat{h}(x)\right)\right]. \tag{C1}$$

Here, ω is the scattering material albedo, $\tau = \alpha L$ is its optical thickness, α is the scattering cross section, and $\hat{h}(x)$ is the Fourier transform of the single scattering phase function: $\hat{h}(x) = \mathcal{F}^{\theta \to x} h(\theta)$. For example, $h(\theta)$ may take the form of the well known Henyey-Greenstein phase function:

$$h(\theta) = \frac{1-g^2}{(1+g^2-2g\cos\theta)^{3/2}}, \tag{C2}$$

where g is the anisotropy (asymmetry) parameter. Alternatively, Mie theory can lead to and exact solution for $h(\theta)$. For example, to create the curve in Figure 3f, we insert the closed-form Mie scattering solution of $h(\theta)$ for a homogeneous sphere (from [Boh1983]) into Eq. (C1).

If we assume the k-space intensity propagator is shift-invariant in k-space, then we may express its one-dimensional form [see Eq. (4) and Eq. (A12)] as, $P_k(\Delta k) = I_k(k_b)$. Shift invariance is not required to connect the SAA with the anisotropic memory effect, but greatly simplifies all relations. Eq. (C1)-Eq. (C2) now directly connect the anisotropic memory effect, through the Fourier transform of the k-space intensity propagator $P_k(\Delta k)$, to material parameters like optical thickness (τ) and anisotropy (g).

A multiplicative relationship between sample optical thickness τ and the shift/shift correlation function C_x directly follows from Eq. (C1). For a slab of thickness 2τ, we first assume shift-invariance to write, $P_k(\Delta k, 2\tau) \propto \mathcal{F}^{x \to \Delta k} \exp\left[-2\tau\left(1 - \omega \hat{h}(x)\right)\right]$. This

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can be equivalently expressed as the Fourier transform of a product: $P_k(\Delta k, 2\tau) \propto \mathcal{F}^{x\to\Delta k}\left[\hat{P}_k(x,\tau)\cdot\hat{P}_k(x,\tau)\right]$. Here, $\hat{P}_k(x,\tau)$ is the Fourier transform of the intensity propagator for a slab of thickness τ: $\hat{P}_k(x,\tau) = \mathcal{F}^{\Delta k\to x}P_k(\Delta k,\tau)$. Since our shift/shift correlation function is also the Fourier transform of P_k, we thus conclude that,

$$P_k(\Delta k, 2\tau) \propto \mathcal{F}^{\Delta x\to\Delta k}[C_x(\Delta x,\tau)\cdot C_x(\Delta x,\tau)]. \quad \text{(C3)}$$

Equivalently, we may remove the Fourier transform to find,

$$C_x(\Delta x, 2\tau) = C_x(\Delta x,\tau)\cdot C_x(\Delta x,\tau). \quad \text{(C4)}$$

We apply the multiplicative relationship in Eq. (C4) to compute the curve shown in Figure 3e of the main text.

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